Classicalisation of Foams in Normed Spaces

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1. Introduction

Swiss cheeses are subsets of the complex plane, introduced by Alice Roth in the 1930s to prove certain results in the field of rational approximation. A Swiss cheese consists of a closed disc with a sequence of open discs removed from its interior such that the resulting set has no interior, but positive area. We call this sequence of open discs a *foam*, which, more generally, is a countable set of open balls. Swiss cheeses whose associated foams have few or no overlaps between discs have many desirable topological properties [1]. This project aims to extend the notion of a foam to normed spaces and introduce a notion of "coalescence" of two balls. We then use this to prove that a foam with overlaps between balls can be turned into a "desirable" foam using such a procedure, for any suitable foam in any Banach space, while preserving certain properties of the original foam.

2. Foams

In general, a foam is a sequence of open balls $((B_i))_{i=1}^{\infty}$, with $B_i \subseteq \mathcal{N}$, $\mathcal{N}$ being a normed space. We denote an open ball with centre y and radius t as $B(y,t)$. We let the closed ball of radius 0 be a point consisting of the centre, and the open ball of radius zero be the empty set.

Semi-classical foams:
A foam such that $B_i \cap B_j = \emptyset$ for all $i \neq j \in \mathbb{N}$.
i.e., no open balls overlap, but may "touch".

Classical foams:
A foam such that $\overline{B_i} \cap \overline{B_j} = \emptyset$ for all $i \neq j \in \mathbb{N}$.
i.e., none of the open balls overlap or touch.

Finite radius sum foams:
We define the *radius sum function,* $\rho: F \mapsto \mathbb{R}$ given by:

$$\rho(F) = \sum_{i=1}^{\infty} r(B_i)$$

Where $F \subseteq \mathcal{N}$ and $r\big(B(y,t)\big) = t$. A foam F with $\rho(F) < \infty$ is a *finite radius sum foam*. We work primarily with finite radius sum foams.
Our aim is to turn a finite radius sum foam into a semi-classical or classical foam, via a series of coalescence operations, while ensuring the radius sum doesn't increase and the resulting union contains the original union. When applied to a finite radius sum Swiss cheese this means that if we replace the original foam the interior will still be empty, and the radius sum of the Swiss cheese will remain finite.

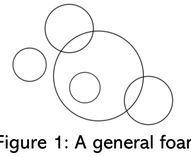


Figure 1: A general foam

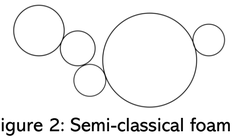


Figure 2: Semi-classical foam

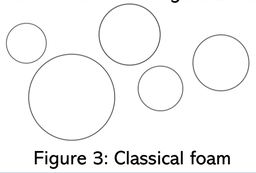


Figure 3: Classical foam

3. Coalescence

Given two overlapping or touching balls in a normed space, say $B(x,r)$ and $B(y,t)$, there exists at least one point, say z such that we can find a ball $B(z,s)$ with:

1. $B(x,r) \subseteq B(z,s)$ and $B(y,t) \subseteq B(z,s)$
2. $s \leq r + t$ ($<$ if they overlap, $=$ if just touching)

We call such a ball a "ball of coalescence" or "coalesced ball". Note that given a pair of overlapping balls there may be no unique ball of coalescence. The uniform norm, for example, would allow for infinitely many choices of coalesced ball. However, there is a unique point on the line between centres that admits a coalesced ball with minimum radius. If we specify that we choose this ball, then we avoid the Axiom of Choice while coalescing.

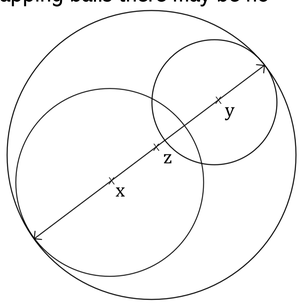


Figure 4: Coalescence

4. The Coalescence & Index Maps

Let $F = F_0$ be some finite radius sum foam in a complete normed space. We create a sequence of foams, $((F_k))_{k=0}^{\infty}$, with each F_{k+1} obtained from F_k by coalescing a pair of balls, say $B_{k,m}$ and $B_{k,n}$ with $m < n$, where $B_{k,i}$ is the i^{th} ball in the k^{th} foam. We make $B_{k+1,m}$ the coalesced ball and $B_{k+1,n}$ empty. This is to ensure no balls disappear in the limit, we call this the "coalescence map". We then define the "index map", which tells us where each index is mapped:

$$f_k\colon \mathbb{N} \mapsto \mathbb{N} \text{ with}$$

$$f_k(t) = \begin{cases} t, \text{if } k = 0 \text{ or } t \neq m, n \\ m, \qquad \text{if } t = m, n \end{cases}$$

where $k, t, m, n \in \mathbb{N}$ and $m < n$. The index of the foam we are mapping to is k. We represent an index in the $k-1^{th}$ foam by t, whilst m and n are the indices of the balls being coalesced in the $k-1^{th}$ foam.
We can then define a map for indices from the initial foam to any later foam:

$$g_k\colon \mathbb{N} \mapsto \mathbb{N} \text{ with}$$

$$g_k(t) = \begin{cases} t, \text{if } k = 0 \\ f_k \circ f_{k-1} \dots \circ f_0(t), \text{if } k \neq 0 \end{cases}$$

where $k, t \in \mathbb{N}$ and t is now an index in F_0. This maps an index from the initial foam to the k^{th} foam and is also non-increasing for a given t as k increases. Observe that $t \geq 0$, and so as $k \to \infty$, $g_k(t)$ must reach a minimum value. Note that for a given i, $B_{0,i} \subseteq B_{k,g_k(i)}\ \forall k \in \mathbb{N}$.

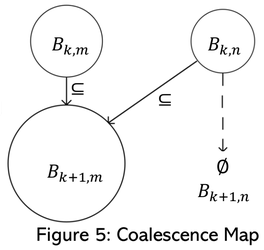


Figure 5: Coalescence Map

5. Limit foams, Semi-classicalisation & Classicalisation

In §4 we saw what a sequence of foams means. We can also think of a *limit foam*, the foam which this sequence of foams tends to. To do this, we define each B_i in our limit foam as the limit of the $B_{k,i} \in F_k$ as $k \to \infty$. We see that for a fixed i, $B_{k,i}$ is eventually empty, or forms a bounded, non-decreasing nested sequence of open bubbles as $k \to \infty$. This can also be extended to limit ordinals.

Semi-classicalisation:
Given any finite radius sum foam, $F = F_0$, in a Banach space, we showed that it is possible to turn this into a semi-classical foam, by the first non-zero limit ordinal, ω, using the coalescence procedure set out in **§4.** We coalesce if there are any overlaps in our foam. If, at some step, there are no overlaps then our sequence of foams is constant from that step onwards. F_ω always has the following properties:

1. $\rho(F_\omega) \leq \rho(F_0)$
2. $\bigcup_{i=1}^{\infty} B_{0,i} \subseteq \bigcup_{i=1}^{\infty} B_{\omega,i}$

However, an arbitrary choice of coalescence cannot be made if you want to be semi-classical by ω. One possible choice is to coalesce the earliest occurrence of the *largest overlap* present in your foam at each step. We know there will always be a maximum size of overlap, as our foam has finite radius sum. Alternatively, among those pairs of indices of balls that overlap, (i,j), with $i \neq j$, you could find and coalesce the pair with $\min\{\max\{(i,j)\}\}$. This guarantees that by some finite step there will be no overlaps present below an index n. If a sensible choice of which balls to coalesce is made at each stage, then F_ω will be *semi-classical*

Classicalisation:
We then introduced ordinals to tackle classicalisation. We showed that given any finite radius sum foam in a complete normed space, we can create a classical foam using the process laid out in **§4**, by some countable ordinal α, i.e., before ω_1, the first uncountable ordinal. This foam, F_α, has the following properties:

1. $\rho(F_\alpha) \leq \rho(F_0)$
2. $\bigcup_{i=1}^{\infty} B_{0,i} \subseteq \bigcup_{i=1}^{\infty} B_{\alpha,i}$
3. F_α is *classical*

Furthermore, $\rho(F_\alpha) = \lim_{\beta \to \alpha} \rho(F_\beta), \beta < \alpha$ (this is also true for semi-classicalisation by ω), and unlike semi-classicalisation an arbitrary choice of coalescence can be made at each step. We created several arguments to show this, the simplest being analogous to one created by J.W.D Mason in [2]. Create a set, Z, containing the indices of those balls "set to 0", simply those balls that are the second of a pair of coalesced balls. Clearly $Z \subseteq \mathbb{N}$. If we kept coalescing to $\omega_1$then Z would contain uncountably many elements, a contradiction. Thus, we always reach a classical foam by some countable ordinal. Another argument focuses on the *squared radius sum*, the sum over i of $r(B_{\beta,i})^2$. Our radius sum function is non-increasing and bounded below as we step through our ordinal sequence of foams, and so is eventually constant. Once this is the case, the squared radius sum function is a non-decreasing function and bounded above by $(\rho(F_0))^2$ so it must eventually be constant. At this point your foam is guaranteed to be classical.
All of this can be independent of the Axiom of Choice, as you can completely specify which balls to coalesce at every stage.
To see why we need completeness, we refer to our definition of a limit foam to see that we have nested non-decreasing sequences of balls, with a bound on the radius. In fact, the sequence of centres of these balls forms a Cauchy sequence, and so we need completeness to ensure the limit ball is still in our normed space at the end of the process.

6. Foams as sets

We now define a foam as a set of open balls, a set of sets. This means that our open balls have no indices associated with them and we do not have any repeats in our foam.
It proved much more challenging to prove the same results on semi-classicalisation and classicalisation in this setting, as a key tool in proving the results in **§5** was the use of *partial radius sums*, i.e., $\sum_{i=1}^{N} r(B_i)$ and $\sum_{i=N+1}^{\infty} r(B_i)$. Instead, we must work with sums over subsets of our foam.
There are some key differences in our results here. We only have $\rho(F_\alpha) \leq \lim_{\beta \to \alpha} \rho(F_\beta), \beta < \alpha$, rather than equality. We can also have three balls mapped to a single ball between foams, whereas in the indexed case we only ever have at most two balls mapped to a single ball. This is because we cannot have repeats in our foams. Say we coalesce two balls, resulting in a third ball already in our foam. Then all three of these balls will be mapped to the coalesced ball, despite one of them not being involved in any coalescence.

7. Reflection

As my first foray into research, this project has given me great insight into the world of academia. I have delved into novel areas of mathematics, including ordinals, metric & normed spaces and measure theory. By writing up our results I have greatly improved my LaTeX skills and learnt how to write up formal mathematical material. The project has convinced me to tackle a master's degree and opened my eyes to the possibility of a PhD. We also hope to submit one or more papers to refereed journals based on this work.

8. References

[1]: J.F. Feinstein, S. Morley, H. Yang, Abstract Swiss cheese space and classicalisation of Swiss cheeses, Journal of Mathematical Analysis and Applications, Volume 438, Issue 1, June 2016 Pages 119-141
[2]: J. W. D. Mason, An inductive proof of the Feinstein-Heath Swiss Cheese "classicalisation" theorem, Proceedings of the American Mathematical Society Volume 138, Number 12, December 2010, Pages 4423–4432, S 0002-9939(2010)10447-1